ON THE LUSTERNIK-SCHNIRELMANN CATEGORY OF SPACES WITH 2-DIMENSIONAL FUNDAMENTAL GROUP

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ABSTRACT. The following inequality

$$\operatorname{cat}_{\mathrm{LS}} X \leq \operatorname{cat}_{\mathrm{LS}} Y + \left\lceil \frac{hd(X) - r}{r+1} \right\rceil$$

holds for every locally trivial fibration between ANE spaces $f : X \to Y$ which admits a section and has the r-connected fiber where $hd(X)$ is the homotopical dimension of X. We apply this inequality to prove that

$$\operatorname{cat}_{\mathrm{LS}} X \leq \left\lceil \frac{\dim X - 1}{2} \right\rceil + cd(\pi_1(X))$$

for every complex X with $cd(\pi_1(X)) \leq 2$.

1. Introduction

In [DKR] we proved that if the Lusternik-Schnirelmann category of a closed n-manifold, $n \geq 3$, equals 2 then the fundamental group of M is free. In opposite direction we proved that if the fundamental group of an n-manifold is free, then $\operatorname{cat}_{\mathrm{LS}} M \leq n-2$. Then J. Strom proved that $\operatorname{cat}_{\mathrm{LS}} X \leq \frac{2}{3}n$ for every n-complex, $n > 4$, with free fundamental group [St]. Yu. Rudyak suggested that the coefficient 2/3 in Strom's result should be improved to 1/2. Precisely, he conjectured that the function f defined as $f(n) = \max\{\operatorname{cat}_{\mathrm{LS}} M^n\}$ is asymptotically $\frac{1}{2}n$ where the maximum is taken over all closed n-manifolds with free fundamental group.

In this paper we prove Rudyak's conjecture. Our method gives the same estimate for n-complexes. Moreover, it gives the same asymptotic upper bound for $\operatorname{cat}_{\mathrm{LS}}$ of n-complexes with the fundamental group of cohomological dimension ≤ 2. In view of this the following generalization of Rudyak's conjecture seems to be natural.

CONJECTURE. *For every k the function f_k defined as*

$$f_k(n) = \max\{\operatorname{cat}_{\mathrm{LS}} M^n \mid cd(\pi_1(M^n) \leq k\}$$

is asymptotically $\frac{1}{2}n$.

The smallest k when it is unknown is 3.

The paper is organized as follows. Section 2 is an introduction to the Lusternik-Schnirelmann category based on an analogy with the dimension theory. Section 3 contains a version of a fibration theorem for $\mathrm{cat}_{\mathrm{LS}}$. In Section 4 this fibration theorem is applied for the proof of Rudyak's conjecture.

2. Kolmogorov-Ostrand's approach to the Lusternik-Schnirelmann category

An open cover $\mathcal{U} = \{U_i\}$ of a topological space X is called *X-contractible* if each U_i can be contracted to a point in X. By the definition $\mathrm{cat}_{\mathrm{LS}} X \le n$ if there is an X-contractible cover of X by $n+1$ elements.

We recall [CLOT] that a sequence $\emptyset = O_0 \subset O_1 \subset \cdots \subset O_{n+1} = X$ is called *categorical of length* $n+1$ if each difference $O_{i+1} \setminus O_i$ is contained in an open set that contracts to a point. It was proven in [CLOT] that $\mathrm{cat}_{\mathrm{LS}} X \le n$ if and only if X admits a categorical sequence of length $n+1$.

Let $\mathcal{U}$ be a family of open set in a topological space X. The multiplicity of $\mathcal{U}$ (or the order) at a point $x \in X$, $\mathrm{Ord}_x \mathcal{U}$ is the number of elements of $\mathcal{U}$ that contain x. The multiplicity of $\mathcal{U}$ is defined as $\mathrm{Ord}\,\mathcal{U} = \sup_{x \in X} \mathrm{Ord}_x \mathcal{U}$. We recall that *the covering dimension of a topological space X does not exceed n, $\dim X \le n$, if for every open cover $\mathcal{C}$ of X there is an open refinement $\mathcal{U}$ with* $\mathrm{Ord}\,\mathcal{U} \le n+1$.

The following proposition makes the LS-category analogous to the covering dimension.

2.1. **Proposition.** *For a paracompact topological space X,* $\mathrm{cat}_{\mathrm{LS}} X \le n$ *if and only if X admits an X-contractible locally finite open cover $\mathcal{V}$ with $Ord\mathcal{V} \le n+1$.*

Proof. If $\mathrm{cat}_{\mathrm{LS}} X \le n$ then by the definition X admits an open contractible cover that consists of $n+1$ elements and therefore its multiplicity is at most $n+1$.

Let $\mathcal{V}$ be a contractible cover of X of multiplicity $\le n+1$. We construct a categorical sequence $O_0 \subset O_1 \subset \cdots \subset O_{n+1}$ of length $n+1$. We define $O_1 = \{x \in X \mid \mathrm{Ord}_x \mathcal{V}\} = n+1$. Note that $O_1 = \cup_{V_i \in \mathcal{V}} V_0 \cap \cdots \cap V_n$. Note that this is the disjoint union and every nonempty summand is X-contractible. Thus O_1 is X-contractible. Next, we define $O_2 = \{x \in X \mid \mathrm{Ord}_x \mathcal{V} \ge n\}$. Then $O_2 \setminus O_1 = cup_{V_i \in \mathcal{V}}(V_1 \cap \cdots \cap V_n \setminus O_1)$ is a disjoint union of closed in O_2 subsets. Since this family is locally finite, we can take open (in O_2 and hence in X) disjoint neighborhoods of these summands $V_1 \cap \cdots \cap V_n \setminus O_1$ such that each of

which is contained in an open contractible set from $\mathcal{V}$. Define $O_3 = \{x \in X \mid Ord_x \geq n-1\}$ as the union of $n-1$-fold intersections and so on. O_{n+1} is the union of elements of $\mathcal{V}$ (1-fold intersections) and hence $O_{n+1} = X$. Clearly, for every k, $O_{k+1} \setminus O_k$ is the union of disjoint sets each of which is contained in an element of $\mathcal{V}$. Thus, the categorical sequence conditions are satisfied. □

A family $\mathcal{U}$ of subsets of X is called a k-cover, $k \in N$ if every subfamily of k elements form a cover of X. The following is obvious.

2.2. **Proposition.** *A family $\mathcal{U}$ of m-elements is an $(n+1)$-cover of X if and only if* $\mathrm{Ord}_x\mathcal{U} \geq m-n$ *for all* $x \in X$.

Proof. If $\mathrm{Ord}_x\mathcal{U} < m-n$ for some $x \in X$, then $n+1 = m-(m-n)+1$ elements of $\mathcal{U}$ do not cover x.

If $n+1$ elements of $\mathcal{U}$ do not cover some x, then $\mathrm{Ord}_x\mathcal{U} \leq m-(n+1) < m-n$. □

Inspired by the work of Kolmogorov on Hilbert's 13 problem Ostrand gave the following characterization of the covering dimension [Os].

2.3. **Theorem** (Ostrand). *A metric space X is of dimension $\leq n$ if and only if for each open cover $\mathcal{C}$ of X and each integer $m \geq n+1$, there exist m discrete families of open sets $\mathcal{U}_1, \cdots, \mathcal{U}_k$ such that their union $\cup\mathcal{U}_i$ is an $(n+1)$-cover of X.*

Let $\mathcal{U}$ be a family of subsets in X and let $A \subset X$. We denote by $\mathcal{U}|_A - \{U \cap A \mid U \in \mathcal{U}\}$.

2.4. **Definition.** Let $f : X \to Y$ be a map. An open cover $\mathcal{U} = \{U_0, U_1, \dots, U_n\}$ of X is called *uniformly f-contractible* if it satisfies the property that for every $y \in Y$ there is a neighborhood V such that the restriction $\mathcal{U}|_{f^{-1}(V)}$ of $\mathcal{U}$ to the preimage $f^{-1}(V)$ consists of X-contractible sets.

2.5. **Theorem.** *Let $\{U_0', \dots, U_n'\}$ be an open cover of a normal topological space X. Then there is an infinite open $(n+1)$-cover of X, $\{U_k\}_{k=0}^{\infty}$ such that $U_k = U_k'$ for $k \leq n$ and $U_k = \cup_{i=0}^{n} V_i$ is the disjoint union with $V_i \subset U_i$ for $k > n$.*

In particular, if $\{U_0', \dots, U_n'\}$ is X-contractible, the cover $\{U_k\}_{k=0}^{\infty}$ is X-contractible. If $\{U_0', \dots, U_n'\}$ is uniformly f-contractible for some $f : X \to Y$, the cover $\{U_k\}_{k=0}^{\infty}$ is uniformly f-contractible.

Proof. By induction on m we construct the family $\{U_i\}_{i=0}^{m}$. For $m = n$ we take $U_i = U_i'$.

Let $\mathcal{U} = \{U_0, \dots, U_m\}$ be the corresponding family for $m \geq n$. Consider $Y = \{x \in X \mid \mathrm{Ord}_x\mathcal{U} \leq m-n\}$. In view of Proposition 2.2 and

the assumption, it is a closed subset. We show that for every $i \leq n$, the set $Y \cap U_i$ is closed in X. Let x be a limit point of $Y \cap U_i$ that does not belong to U_i. Let $U_{i_0}, \dots, U_{i_{m-n}}$ be the elements that contain x. Then $\mathrm{Ord}_y \mathcal{U} = m - n + 1$ for all $y \in Y \cap U_i \cap U_{i_0} \cap \dots \cap U_{i_{m-n}}$. Contradiction.

We define recursively $F_0 = Y \cap U_1$ and $F_{i+1} = Y \cap U_{i+1} \setminus (\cup_{k=0}^{i} U_i)$. Note that $\{F_i\}$ is a disjoint finite family of closed subsets. We fix disjoint open neighborhoods V_i of F_i with $V_i \subset U_i$. We define $U_{m+1} = \cup_i V_i$. In view of Proposition 2.2, $U_1, \dots, U_m, U_{m+1}$ is an $(n+1)$-cover.

Clearly, if all U_i are X-contractible, $i \leq n$, then U_{m+1} is X-contractible. If all U_i are uniformly f-contractible, for some $f : X \to Y$, then U_{m+1} is uniformly f-contractible. □

2.6. **Corollary.** For a normal topological space X, $\mathrm{cat}_{\mathrm{LS}} X \leq n$ if and only if for any $m > n$ X admits an open $(n+1)$-cover by X-contractible sets.

This corollary is a $\mathrm{cat}_{\mathrm{LS}}$-analog of Ostrand's theorem. It also can be found in [CLOT] with further reference to [Cu].

3. Fibration theorems for $\mathrm{cat}_{\mathrm{LS}}$

3.1. **Definition.** The $*$-category $\mathrm{cat}_{\mathrm{LS}}{}^* f$ of a map $f : X \to Y$ is the minimal n, if exists, such that there is a uniformly f-contractible open cover $\mathcal{U} = \{U_0, U_1, \dots, U_n\}$ of X.

Note that $\mathrm{cat}_{\mathrm{LS}}{}^* c = \mathrm{cat}_{\mathrm{LS}} X$ for a constant map $c : X \to pt$. More generally, $\mathrm{cat}_{\mathrm{LS}}{}^* \pi = \mathrm{cat}_{\mathrm{LS}} X$ for the projection $\pi : X \times Y \to Y$.

3.2. **Theorem.** *The inequality* $\mathrm{cat}_{\mathrm{LS}} X \leq \dim Y + \mathrm{cat}_{\mathrm{LS}}{}^* f$ *holds true for any continuous map.*

Proof. The requirements to the spaces in the theorem are that the Ostrand theorem holds true for Y, i.e. are fairly general (say, Y is normal).

Let $\dim Y = n$ and $\mathrm{cat}_{\mathrm{LS}}{}^* f = m$. Let $\mathcal{U} = \{U_0, \dots U_m\}$ be a uniformly f-contractible cover. For all $y \in Y$ denote by V_y a neighborhood of y from the definition of the uniform f-contractibility. In view of Ostrand's characterization of dimension there is a refinement $\mathcal{V} = \mathcal{V}_0 \cup \dots \cup \mathcal{V}_{n+m}$ of the cover $\{V_y \mid y \in Y\}$ such that each family $\mathcal{V}_i$ is disjoint. Let $V_i = \cup \mathcal{V}_i$. We apply Theorem 2.5 to extend the family $\mathcal{U}$ to a family $\{U_0, \dots, U_{n+m}\}$. Consider the family $\mathcal{W} = \{f^{-1}(V_i) \cap U_i\}$. Note that it is contractible in X. We show that it is a cover of X. According to the Ostrand theorem every $y \in Y$ is covered by $m + 1$ elements $V_{i_0}, \dots, V_{i_m}$. By Theorem 2.5 the family $U_{i_0}, \dots, U_{i_m}$ covers the fiber $f^{-1}(y)$. □

3.3. **Corollary** (Corollary 9.35 [CLOT], [OW])**.** Let $p : X \to Y$ be a closed map of ANE. If each fiber $p^{-1}(y)$ is contractible in X, then $\operatorname{cat}_{\mathrm{LS}} X \leq \dim Y$.

Proof. In this case $\operatorname{cat}_{\mathrm{LS}}{}^* p = 0$. Indeed, since X is an ANE, a contraction of $p^{-1}(y)$ to a point can be extended to a neighborhood U. Since the map p is closed there is a neighborhood V of y such that $p^{-1}(V) \subset U$. □ □

We recall that the *homotopical dimension* of a space X, $hd(X)$, is the minimal dimension of a CW-complex homotopy equivalent to X [CLOT].

The elementary obstruction theory implies the following.

3.4. **Proposition.** *Let $p : E \to X$ be a fibration with $n-1$-connected fiber where $n = hd(X)$. Then p admits a section.*

We recall that $\lceil x \rceil$ denotes the smallest integer n such that $x \leq n$.

3.5. **Theorem.** *Suppose that a locally trivial fibration $f : X \to Y$ with an r-connected fiber F admits a section. Then*

$$\operatorname{cat}_{\mathrm{LS}}{}^* f \leq \left\lceil \frac{hd(X) - r}{r+1} \right\rceil .$$

Moreover,

$$\operatorname{cat}_{\mathrm{LS}} X \leq \operatorname{cat}_{\mathrm{LS}} Y + \left\lceil \frac{hd(X) - r}{r+1} \right\rceil .$$

Proof. Let $\operatorname{cat}_{\mathrm{LS}} Y = m$ and $hd(X) = n$.

Let $s : Y \to X$ be a section. We apply the fiber-wise Serre construction to f with this $s(y)$ as the base points. Then we apply the fiber-wise Ganea construction to obtain a map $\xi_k : E_k \to X$. Namely,

$$E_0 = \{\phi \in X^I \mid |f(\phi(I))| = 1,\ \phi(0) = sf\phi(I)\}$$

with $\xi_0(\phi) = \phi(1)$ and

$$E_{k+1} = \{\phi * \psi \in E_0 * E_k \mid \xi_0(\phi) = \xi_k(\psi)\}$$

with the natural projection $\xi_{k+1} : E_{k+1} \to X$. Note that ξ_k is a Hurewicz fibration with the fiber the join product $*^{k+1}\Omega F$ of $k+1$ copies of the loop space ΩF. Thus, it is $(k+(k+1)r-1)$-connected. By Proposition 3.4 there is a section $\sigma : X \to E_k$ whenever $k(r+1)+r \geq n$. The smallest such k is equal to $\lceil \frac{gcd(X)-r}{r+1} \rceil$.

For each $x \in X$ the element$\sigma(x)$ of $*_k\Omega F$ can be presented as the $(k+1)$-tuple

$$\sigma(x) = ((\phi_0, t_0), \ldots, (\phi_k, t_k)) \mid \sum t_i = 1, t_i \geq 0).$$

We use the notation $\sigma(x)_i = t_i$. Clearly, $\sigma(x)_i$ is a continuous function.

A section $\sigma : X \to E_k$ defines a cover $\mathcal{U} = \{U_0, \dots, U_k\}$ of X as follows:

$$U_i = \{x \in X \mid \sigma(x)_i > 0\}.$$

Let $\{U_0, \dots, U_{m+k}\}$ be an extension of $\mathcal{U}$ to an $(k+1)$-cover of X from Theorem 2.5.

Let $\mathcal{V} = \{V_0, \dots V_{m+k}\}$ be an open Y-contractible $(m+1)$-cover of Y. We show that the sets $W_i = f^{-1}(V_i) \cap U_i$ are contractible in X for all i. By the construction of U_i for $i \le n$ for every $x \in U_i$ there is a canonical path connecting x with $sf(x)$. We use these paths to contract $f^{-1}(V_i) \cap U_i$ to $s(V_i)$ in X. Then we apply a contraction of $s(V_i)$ to a point in $s(Y)$. If $i > k$, we apply the last condition of the first part of Theorem 2.5.

Similarly as in the proof of Theorem 3.2 we show that $\{W_i\}_{i=0}^{m+k}$ is a cover of X. Since $\mathcal{V}$ is an $(m+1)$-cover, by Proposition 2.2 every $y \in Y$ is covered by at least $k+1$ elements $V_{i_0}, \dots, V_{i_k}$ of $\mathcal{V}$. By the construction $U_{i_0}, \dots, U_{i_k}$ is a cover of X. Hence $W_{i_0}, \dots, W_{i_k}$ covers $f^{-1}(y)$. □

4. The Lusternik-Schnirelmann category of complexes with low dimensional fundamental groups

4.1. **Theorem.** *For every complex X with $cd(\pi_1(X)) \le 2$ the following inequality holds true:*

$$\operatorname{cat}_{\mathrm{LS}} X \le cd(\pi_1(X)) + \left\lceil \frac{hd(X) - 1}{2} \right\rceil .$$

Proof. Let $\pi = \pi_1(X)$. We consider Borel's construction

$$\begin{array}{ccccc} \tilde{X} & \longleftarrow & \tilde{X} \times E\pi & \longrightarrow & E\pi \\ \downarrow & & \downarrow & & \downarrow \\ X & \xleftarrow{g} & \tilde{X} \times_\pi E\pi & \xrightarrow{f} & B\pi. \end{array}$$

We claim that there is a section $s : B\pi \to \tilde{X} \times_\pi E\pi$ of f. By the condition $cd\pi \le 2$ we may assume that $B\pi$ is a complex of dimension ≤ 3. Since a fiber of f is simply connected, there is a lift of the 2-skeleton. The condition $cd\pi \le 2$ implies $H^3(B\pi, E) = 0$ for every π-module. Thus, we have no obstruction for the lift of the 3-skeleton.

We apply Theorem 3.5 to obtain the inequality

$$\operatorname{cat}_{\mathrm{LS}} X \le \operatorname{cat}_{\mathrm{LS}}(B\pi) + \left\lceil \frac{hd(\tilde{X} \times_\pi E\pi) - 1}{2} \right\rceil .$$

Since g is a fibration with the homotopy trivial fiber, the space $\tilde{X}\times_\pi E\pi$ is homotopy equivalent to X. Thus, $hd(\tilde{X}\times_\pi E\pi) = hd(X)$. Note that $\operatorname{cat}_{\mathrm{LS}} B\pi = cd\pi$. □

4.2. **Corollary.** For every complex X with free fundamental group:

$$\operatorname{cat}_{\mathrm{LS}} X \le 1 + \left\lceil \frac{\dim X - 1}{2} \right\rceil.$$

Note that this estimate is sharp on $X = S^1 \times \mathbb{C}P^n$.

4.3. **Corollary.** For every 3-dimensional complex X with free fundamental group: $\operatorname{cat}_{\mathrm{LS}} X \le 2$.

This Corollary can be also derived from the fact that in the case of free fundamental group every 2-complex is homotopy equivalent to the wedge of circles and 2-spheres [KR].

It is unclear whether the estimate $\operatorname{cat}_{\mathrm{LS}} X \le 2 + \lceil \frac{\dim X - 1}{2} \rceil$ is sharp for complexes with $cd(\pi_1(X)) = 2$. It is sharp when the answer to the following question is affirmative.

4.4. **Question.** Does there exists a 4-complex K with free fundamental group and with $\operatorname{cat}_{\mathrm{LS}}(K \times S^1) = 4$?

Indeed, for $X = K \times S^1$ we would have the equality $4 = 2 + \lceil \frac{5-1}{2} \rceil$. Note that $cd(\pi_(X)) = 2$. There is a connection between this question and the Problem 1.5 from [DKR] which asks whether $\operatorname{cat}_{\mathrm{LS}} M^4 \le 2$ for closed 4-manifolds with free fundamental group.

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